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PERTURBATION METHOD FOR HEAT EXCHANGE BETWEEN A GAS AND SOLID PARTICLES

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The analytical perturbation method is applied here to solve the problem of radiative heat transfer between a gas and solid particles. The data obtained are compared with results calculated by the numerical Runge–Kutta method.

Key words: *perturbation method, radiation, dust flame, particle temperature.*

Introduction. Most of engineering problems are solved with the use of nonlinear heat-transfer equations, and it is difficult to solve them analytically [1, 2]. In recent decades, numerical calculation methods were good means of analyzing such equations; improvement of numerical methods led to development of approximate analytical methods [3–11].

The perturbation method is an approximate method used to solve nonlinear equations, which was studied by a large number of researchers (see, e.g., [12–14]). In these publications, however, more attention was paid to mathematical aspects of the subject. Physical verification was provided in [15, 16].

1. Mathematical Modeling. In this modeling, a single solid particle is heated from ambient conditions to the ignition point in the preheat zone. A nonlinear heat-transfer equation including a nonlinear radiative term is used to describe this process. The solid particle temperature T_s is determined as a function of the particle size, specific heat, and flame temperature; it is assumed to be equal to the ambient temperature (300 K). The lumped capacitance and the particle temperature are assumed to be described by the transient energy equation

$$m_s c_s \frac{dT_s}{dt} = q_{\rm conv} + q_{\rm rad}, \qquad (1)$$

where $q_{\rm conv}$ and $q_{\rm rad}$ are the convective and radiative heat fluxes from the ambient medium to the particle, respectively. The convective heat flux is given by Newton's law of cooling:

$$q_{\rm conv} = h(T_g - T_s). \qquad (2)$$

The convective heat-transfer coefficient is calculated with the use of the Nusselt number correlation for a convective flow over a spherical particle:

$$\mathrm{Nu} = 2 + (0.4\,\mathrm{Re}^{0.5} + 0.06\,\mathrm{Re}^{0.75})\mathrm{Pr}^{\,0.4}(\mu/\mu_s)^{0.25} = hD/\lambda_g, \qquad \mathrm{Re} = V_s D/\nu.$$

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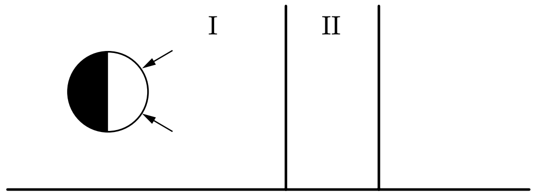

Fig. 1. Heating of the particle by the ambient medium: the preheat zone and the flame zone are indicated by I and II; the arrows show the radiative flux direction.

The thermal properties of the gas are found by the one third rule

$$T_g = T_u/3 + 2T_f/3,$$

where $T_u = 300$ K and $T_f = 2600$ K. It is assumed that the particle velocity is approximately equal to the gas velocity ($V_s = 0$ and $\mathrm{Nu} = 2$) and the particle size is $d_s = 5$ μm. If the complete entrainment assumption is invalid ($V_s \neq 0$), the Stokes flow relation along with the gravitational force is used to find the relative particle velocity (V_s) in the flow.

The net radiative heat flux from the hot ambient medium to the cold particle is given by the Stefan–Boltzmann law

$$q_{\mathrm{rad}} = \varepsilon\sigma A_{\mathrm{eff}} F_{\mathrm{SF}} (T_f^4 - T_s^4), \tag{3}$$

where A_{eff} is the effective area, which is half of the total area, and F_{SF} is the shape factor.

Owing to a negligible difference between the ambient and particle temperatures, the radiative heat transfer between the left area of the particle and the ambient medium is neglected (Fig. 1). Only the radiative heat transfer between the particle and the flame is considered. Substituting the convective and radiative heat fluxes (2) and (3) into Eq. (1), we obtain the rate of temperature variation in time for a spherical particle:

$$m_s c_s \frac{dT_s}{dt} + hA_s(T_s - T_g) + \varepsilon\sigma A_{\mathrm{eff}} F_{sp} (T_s^4 - T_f^4) = 0. \tag{4}$$

Here, $A_s = 4\pi r^2$, $A_{\mathrm{eff}} = \pi r^2$, and $F_{sp} = 1/2$. The gas temperature in the preheat zone is determined by the equation [17]

$$T_g = T_a + (T_f - T_u)\,\mathrm{e}^{\varkappa x/(V_u \tau_c)}, \tag{5}$$

where $\varkappa$ is the dimensionless burning rate. By changing the variable $dx = V_u\,dt$, we obtain the following equation from Eq. (4):

$$V_u m_s c_s \frac{dT_s}{dx} + 4\pi r_s^2 h(T_s - T_g) + \frac{1}{2}\,\varepsilon\sigma\pi r_s^2 (T_s^4 - T_f^4) = 0. \tag{6}$$

To solve Eq. (6), we introduce the following parameters:

$$\theta_s = \frac{T_s}{T_u}, \quad \theta_f = \frac{T_f}{T_u}, \quad \theta_g = \frac{T_g}{T_u}, \quad y = \frac{x}{V_u \tau_c}$$

($V_u \tau_c$ is the flame thickness). As a result, the heat-transfer equation acquires the form

$$\frac{d\theta_s}{dy} + A\theta_s + B\theta_s^4 = A\theta_g(y) + B\theta_f^4, \tag{7}$$

where

$$A = \pi d_s^2 \tau_c/(m_s c_s), \quad B = \varepsilon\sigma\pi d_s^2 T_u^3 \tau_c/(2m_s c_s), \quad \theta_g(y) = 1 + (\theta_f - 1)\,\mathrm{e}^{\varkappa y},$$

and y is the normalized distance between the flame boundary and the particle.

2. Perturbation Method. In this section, we apply the perturbation method to solve Eq. (7). For the weak radiation regime ($B \ll 1$), we calculate the first three terms in the regular expansion of the perturbation:

$$\theta_s = \theta_0 + B\theta_1 + B^2\theta_2 + B^3\theta_3 + \ldots. \tag{8}$$

Inserting Eq. (8) into Eq. (7) and collecting terms based on powers of B as $0, 1, 2, \ldots$, we obtain

$$\begin{aligned}
&B^0: && \frac{d\theta_0}{dy} + A\theta_0 = A\theta_g(y), && -\infty < y \le 0,\\
& && y = -\infty, \qquad \theta_0 = 1, &&\\
&B^1: && \frac{d\theta_1}{dy} + A\theta_1 = \theta_f^4 - \theta_0^4, && -\infty < y \le 0,\\
& && y = -\infty, \qquad \theta_1 = 0, &&\\
&B^2: && \frac{d\theta_2}{dy} + A\theta_2 = -4\theta_0^3\theta_1, && -\infty < y \le 0,\\
& && y = -\infty, \qquad \theta_2 = 0, &&
\end{aligned} \tag{9}$$

etc. The solutions of Eqs. (9) are

$$\begin{aligned}
&\theta_0 = 1 + M\,\mathrm{e}^{\varkappa y},\\
&\theta_1 = \frac{\theta_f^4 - 1}{A} - 4\,\frac{M\,\mathrm{e}^{\varkappa y}}{A+\varkappa} - 6\,\frac{M^2\,\mathrm{e}^{2\varkappa y}}{A+2\varkappa} - 4\,\frac{M^3\,\mathrm{e}^{3\varkappa y}}{A+3\varkappa} - \frac{M^4\,\mathrm{e}^{4\varkappa y}}{A+4\varkappa},\\
&\theta_2 = 4\,\frac{\theta_f^4}{A^2} + \frac{4}{A^2} + 4\,\frac{M^4\,\mathrm{e}^{4\varkappa y}}{(4\varkappa+A)^2} + 16\,\frac{M^3\,\mathrm{e}^{3\varkappa y}}{(3\varkappa+A)^2} + 24\,\frac{M^2\,\mathrm{e}^{2\varkappa y}}{(2\varkappa+A)^2}\\
&\quad + 16\,\frac{M\,\mathrm{e}^{\varkappa y}}{(\varkappa+A)^2} - 12\,\frac{M\theta_f^4\,\mathrm{e}^{\varkappa y}}{A(\varkappa+A)} + 12\,\frac{M\,\mathrm{e}^{\varkappa y}}{A(\varkappa+A)} + 12\,\frac{M^5\,\mathrm{e}^{5\varkappa y}}{(4\varkappa+A)(A+5\varkappa)}\\
&\quad + 48\,\frac{M^4\,\mathrm{e}^{4\varkappa y}}{(3\varkappa+A)(4\varkappa+A)} + 72\,\frac{M^3\,\mathrm{e}^{3\varkappa y}}{(2\varkappa+A)(3\varkappa+A)} + 48\,\frac{M^2\,\mathrm{e}^{2\varkappa y}}{(\varkappa+A)(2\varkappa+A)}\\
&\quad - 12\,\frac{M^2\,\theta_f^4\,\mathrm{e}^{2\varkappa y}}{A(2\varkappa+A)} + 12\,\frac{M^2\,\mathrm{e}^{2\varkappa y}}{A(2\varkappa+A)} + 12\,\frac{M^6\,\mathrm{e}^{6\varkappa y}}{(4\varkappa+A)(A+6\varkappa)}\\
&\quad + 48\,\frac{M^5\,\mathrm{e}^{5\varkappa y}}{(3\varkappa+A)(A+5\varkappa)} + 72\,\frac{M^4\,\mathrm{e}^{4\varkappa y}}{(2\varkappa+A)(4\varkappa+A)} + 48\,\frac{M^3\,\mathrm{e}^{3\varkappa y}}{(\varkappa+A)(3\varkappa+A)}\\
&\quad - 4\,\frac{M^3\theta_f^4\,\mathrm{e}^{3\varkappa y}}{A(3\varkappa+A)} + 4\,\frac{M^3\,\mathrm{e}^{3\varkappa y}}{A(3\varkappa+A)} + 4\,\frac{M^7\,\mathrm{e}^{7\varkappa y}}{(4\varkappa+A)(A+7\varkappa)} + 16\,\frac{M^6\,\mathrm{e}^{6\varkappa y}}{(3\varkappa+A)(A+6\varkappa)}\\
&\quad + 24\,\frac{M^5\,\mathrm{e}^{5\varkappa y}}{(2\varkappa+A)(A+5\varkappa)} + 16\,\frac{M^4\,\mathrm{e}^{4\varkappa y}}{(\varkappa+A)(4\varkappa+A)},
\end{aligned} \tag{10}$$

etc. In these solutions,

$$M = A(\theta_f - 1)/(\varkappa + A).$$

Inserting Eqs. (10) into Eqs. (8), we obtain an approximate temperature distribution of the solid particle.

3. Computation of Radiative Coefficients. In this work, aluminum particles with the following parameters are used: particle density $\rho_s = 2707$ kg/m^3, heat capacity $c_s = 0.896$ kJ/(kg·K), particle diameter $d_s = 5.4$ μm, burning rate $V_u = 0.5$ m/sec, time of particle combustion $\tau_c = 0.1$ msec, and flame temperature $T_f = 3000$ K. The parameters of the ambient medium [temperature $T_u = 300$ K, density $\rho_u = 1.1774$ kg/m^3, and heat capacity $\lambda_u = 0.02624$ W/(m·K)] and of the gas (density $\rho_g = 0.1308$ kg/m^3 and Stefan–Boltzmann constant

TABLE 1

Results of Solutions Obtained by the Perturbation Method
of the Third (PM3) and Fifth (PM5) Orders and by the Runge–Kutta Method,
and the Differences in These Results

y	PM3	PM5	RK45	$\Delta 3$	$\Delta 5$
−9	1.788,400,550	1.788,370,399	1.787,924,124	0.000,476,426	0.000,446,275
−8	1.790,303,334	1.790,273,184	1.789,825,595	0.000,477,739	0.000,447,589
−7	1.795,475,592	1.795,445,445	1.794,993,081	0.000,482,512	0.000,452,365
−6	1.809,534,915	1.809,504,777	1.809,041,614	0.000,493,301	0.000,463,163
−5	1.847,749,585	1.847,719,475	1.847,228,346	0.000,521,240	0.000,491,129
−4	1.951,607,804	1.951,577,783	1.951,003,103	0.000,604,701	0.000,574,680
−3	2.233,747,277	2.233,717,654	2.232,886,685	0.000,860,591	0.000,830,968
−2	2.998,722,224	2.998,695,642	2.996,961,632	0.001,760,592	0.001,734,010
−1	5.046,253,462	5.046,268,507	5.041,532,793	0.004,720,669	0.004,735,714
0	9.988,489,409	9.988,487,246	9.989,971,475	0.001,482,065	0.001,484,228

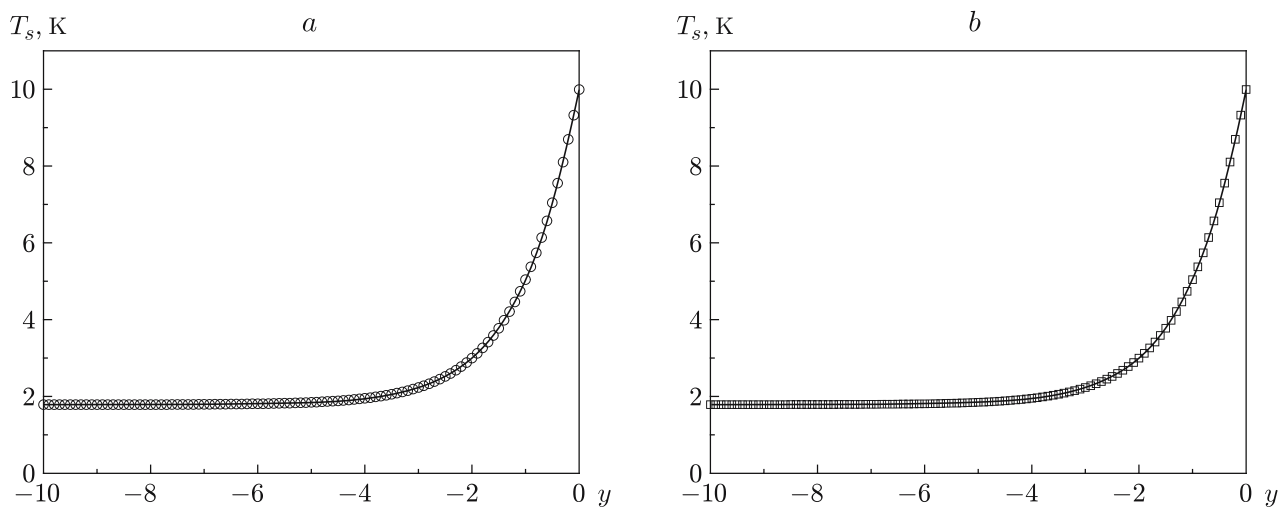

Fig. 2. Particle temperature distribution: (a) third-order perturbation method; (b) fifth-order perturbation method; the points and curves show the results obtained by the perturbation method and the Runge–Kutta method, respectively.

$\sigma = 0.567 \cdot 10^{-7}$) are assumed to be constant. Substituting these values into Eqs. (10), we find the aluminum particle temperature:

$$\theta_0 = 1.0 + 8.9827\,\mathrm{e}^{y},$$

$$\theta_1 = 19.2552\,\mathrm{e}^{519.2886y} - 0.069\,06\,\mathrm{e}^{520.2886y} - 0.9287\,\mathrm{e}^{521.2886y} - 5.551\,\mathrm{e}^{522.2886y} - 12.4419\,\mathrm{e}^{523.2886y},$$

$$\theta_2 = -0.1483 + 3.3396\,\mathrm{e}^{4y} - 106.5231\,\mathrm{e}^{3y} - 35.7441\,\mathrm{e}^{2y}$$
$$- 3.9887\,\mathrm{e}^{y} + 17.9454\,\mathrm{e}^{5y} + 53.5717\,\mathrm{e}^{6y} + 68.5399\,\mathrm{e}^{7y},$$

$$\theta_3 = 1.213{,}756{,}527 \cdot 10^{-36}\,\mathrm{e}^{8y} + 3.921{,}203{,}592 \cdot 10^{-37}\,\mathrm{e}^{3y} + 9.784{,}275{,}514 \cdot 10^{-39}\,\mathrm{e}^{4y}$$
$$+ 6.138{,}633{,}948 \cdot 10^{-36}\,\mathrm{e}^{2y} + 2.436{,}755{,}641 \cdot 10^{-39}\,\mathrm{e}^{6y} + 1.209{,}170{,}157 \cdot 10^{-36}\,\mathrm{e}^{10y}$$
$$+ 1.945{,}700{,}453 \cdot 10^{-37}\,\mathrm{e}^{7y} + 3.851{,}422{,}764 \cdot 10^{-19} + 3.075{,}216{,}233 \cdot 10^{-27}\,\mathrm{e}^{y}$$
$$+ 1.953{,}122{,}702 \cdot 10^{-38}\,\mathrm{e}^{5y} + 3.876{,}668{,}804 \cdot 10^{-35}\,\mathrm{e}^{9y},$$

$$\theta_4 = 2.446{,}068{,}878 \cdot 10^{-73}\, e^{4y} + 2.432{,}125{,}566 \cdot 10^{-85}\, e^{7y} + 3.844{,}020{,}290 \cdot 10^{-49}\, e^{y}$$
$$+ 1.202{,}355{,}225 \cdot 10^{-82}\, e^{13y} + 2.427{,}513{,}053 \cdot 10^{-84}\, e^{8y} + 1.206{,}889{,}946 \cdot 10^{-82}\, e^{11y}$$
$$+ 3.069{,}316{,}974 \cdot 10^{-57}\, e^{2y} + 4.818{,}473{,}272 \cdot 10^{-81}\, e^{12y} + 1.209{,}170{,}157 \cdot 10^{-83}\, e^{10y}$$
$$+ 1.211{,}459{,}001 \cdot 10^{-85}\, e^{9y} + 2.450{,}752{,}244 \cdot 10^{-65}\, e^{3y} + 4.882{,}806{,}755 \cdot 10^{-82}\, e^{5y}$$
$$+ 7.702{,}845{,}527 \cdot 10^{-40} + 9.747{,}022{,}561 \cdot 10^{-83}\, e^{6y}, \tag{11}$$

$$\theta_5 = -5033.667{,}081\, e^{13y} - 9646.796{,}268\, e^{14y} - 6430.119{,}927\, e^{16y} - 11{,}503.483{,}72\, e^{15y}$$
$$- 24{,}986.292{,}36\, e^{8y} + 21{,}823.677{,}71\, e^{12y} + 31{,}236.378{,}17\, e^{11y} + 49.264{,}868{,}50\, e^{y}$$
$$+ 8552.564{,}771\, e^{4y} + 3925.829{,}450\, e^{3y} + 660.711{,}413{,}70\, e^{2y} - 2090.454{,}586\, e^{5y}$$
$$- 9330.130{,}259\, e^{6y} - 23{,}640.060{,}34\, e^{7y} + 7251.921{,}209\, e^{9y} - 263.446{,}745{,}10 + 19{,}405.203{,}19\, e^{10y}\,.$$

Finally, by inserting Eqs. (11) into Eqs. (8), we find the temperature distribution for aluminum particles. The results of solving the heat-conduction equation by the perturbation method of the third and fifth orders are listed in Table 1. The table also contains the differences between the results calculated by the perturbation method of the third (PM3) and fifth (PM5) orders and the results obtained by the Runge–Kutta method (RK45): $\Delta 3 = |\mathrm{PM3} - \mathrm{RK45}|$ and $\Delta 5 = |\mathrm{PM5} - \mathrm{RK45}|$, respectively. It follows from the table that the values obtained by the perturbation method are in good agreement with numerical data, as is illustrated in Fig. 2.

Conclusions. In this paper, the perturbation method of the third and fifth orders is applied to solve the radiative heat-transfer equation. A comparison of the solutions obtained shows the validity and great potential of the perturbation method for solving nonlinear problems. The preheat-zone thickness is found to be approximately twice the flame-zone thickness.

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